

Note on the Diameter of Path-Pairable Graphs

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Abstract

A graph on $2k$ vertices is path-pairable if for any pairing of the vertices the pairs can be joined by edge-disjoint paths. The so far known families of path-pairable graphs have diameter of at most 3. In this paper we present an infinite family of path-pairable graphs with diameter $d(G) = O(\sqrt{n})$ where n denotes the number of vertices of the graph. We prove that our example is extremal up to a constant factor.

Introduction

Given a fixed integer k , a graph G on at least $2k$ vertices is k -path-pairable if for any pair of disjoint sets of vertices $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_k\}$ of G there exist k edge-disjoint paths P_i such that P_i is a path from x_i to y_i , $1 \leq i \leq k$. The path-pairability number of a graph G is the largest positive integer k for which G is k -path-pairable. The motivation of setting edge-disjoint paths between certain pairs of nodes naturally arose in the study of communication networks. There are various reasons to measure the capability of the network by its path-pairability number, that is, the maximum number of pairs of users for which the network can provide separated communication channels without data collision (see [1] for additional details). The nodes corresponding to the users are often called *terminal nodes* or *terminals*. A graph G on $n = 2m$ vertices is *path-pairable* if it is m -path-pairable, that is, for every pairing of the vertex set $\{x_1, y_1, \dots, x_m, y_m\}$ there exist edge-disjoint paths P_1, \dots, P_m joining x_1 to y_1, \dots, x_m to y_m , respectively. By definition, path-pairable graphs are k -path-pairable for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

The three dimensional cube Q_3 and the Petersen graph P are both known to be path-pairable. The graph shown in Figure 1 is the only path-pairable graph with maximum degree 3 on 12 or more vertices. Apart from such small and rather sporadic examples we only know few path-pairable families. Certainly, the complete graph K_{2k} on $n = 2k$ vertices is path-pairable. It can be proved easily that the complete bipartite graph $K_{a,b}$ is path-pairable as long as $a + b$ is even and $a, b \neq 2$. Particular species of the latter family, the star graphs $K_{2a+1,1}$ show that path-pairability is achievable even in the presence of vertices of small degrees. They also illustrate that vertices of large degrees are easily accessible transfer stations to carry out linking in the graphs without much effort. That motivates the study of k -path-pairable and path-pairable graphs with small maximum degree. Faudree, Gyárfás and Lehel [3] gave examples of k -path-pairable graphs for every $k \in \mathbb{N}$ with maximum degree 3. Note that their construction has exponential size in terms of k and is not path-pairable. Unlike in case of

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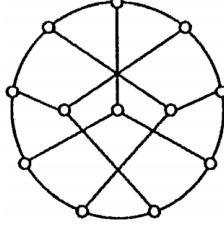


Figure 1: Path-pairable graph of order 12

k -path-pairability, the maximum degree $\Delta(G)$ must increase together with the size of a path-pairable graph G . Faudree, Gyárfás and Lehel [4] proved that if G is a path-pairable graph on n vertices with maximum degree Δ then $n \leq 2\Delta^\Delta$. The theorem gives an approximate lower bound of $\frac{\log(n)}{\log \log(n)}$ on $\Delta(G)$. By contrast, the graphs of the above presented families have maximum degree $\frac{n}{2}$ or more. Kubicka, Kubicki and Lehel [5] investigated path-pairability of complete grid graphs and proved that the two-dimensional complete grid $K_a \times K_b$ of size $n = ab$ is path-pairable. For $a = b$ that gives examples of path-pairable graphs with maximum degree $\Delta = 2a - 2 < 2\sqrt{n}$. In the same paper they raised the question about similar properties of three-dimensional complete grids. Note that if path-pairable, the grid $K_m \times K_m \times K_m$ yields an even better example of size $n = m^3$ and maximum degree $\Delta = 3(m - 1) = O(\sqrt[3]{n})$.

We mention that one of the most interesting and promising path-pairable candidate is the n -dimensional hypercube Q_n . $Q_1 = K_2$ is path-pairable while $Q_2 = C_4$ is not as pairing of the non-adjacent vertices of the cycle cannot be linked. One can prove that the cube Q_3 is path-pairable, the question for higher odd-dimensions has yet to be answered (it was proved in [2] that Q_n is not path-pairable for n even).

Conjecture 1 ([1]). The $(2k + 1)$ -dimensional hypercube Q_{2k+1} is path-pairable for all $k \in \mathbb{N}$.

A common attribute of the known path-pairable graphs is their small diameter. For each pair (x, y) of terminals in the examples above, the length of the shortest x, y paths is at most 3. While terminal pairs of an actual pairing may not always be joined by shortest paths, small diameter gives the advantage of quick accessibility of the vertices and makes designation of edge-disjoint paths easier. The question concerning the existence of an infinite family of path-pairable graphs with unbounded diameter naturally arises. We use the notation $d(G)$ for the diameter of the graph G and $d_{\max}(n)$ for the maximal diameter of a path-pairable graph on n vertices. We mention that if true, Conjecture 1 proves the lower bound $\log_2 n \leq d_{\max}(n)$ for $n = 2^{2k+1}$, $k \in \mathbb{N}$.

The main goal of this note is to study the largest possible diameter of path-pairable graphs. We present a family of path-pairable graphs $\{G_n\}$ such that G_n has n vertices and diameter $O(\sqrt{n})$ for infinitely many values of n . We show that our construction is optimal up to a constant factor by proving the following theorem.

Theorem 1. *If G is a path-pairable graph on n vertices with diameter $d \geq 20$ then $d \leq 6\sqrt{2} \cdot \sqrt{n}$.*

For a subgraph H of a graph G , $|H|$ denotes the number of vertices in H . The degree of a vertex x and the distance of vertices x and y are denoted by $d(x)$ and $d(x, y)$, respectively. For additional details on path-pairable graphs we refer the reader to [1], [2] and [4].

Construction

We construct our example of a path-pairable graph on n vertices with diameter $O(\sqrt{n})$ by the graph operation called "blowing-up".

Set an arbitrary pairing of the vertices of G . We accomplish the linking of the pairs in two phases. During the first phase, for each pair of terminals, we define a path that starts at one of the terminals and ends at some vertex in the class of its pair. If the ending vertex happens to be the actual pair of the terminal, we set this path as the joining path for the given pair, otherwise we continue with the second phase. If two terminals initially belong to the same class, then the pair simply skips the first phase of the linking. Direct our cycle C_{2m} and the blown-up graph G counterclockwise and label each pair x, y such that there exists a directed $x \rightarrow y$ path of length at most m . We start building the above mentioned path for pair (x, y) at vertex x . Fix m edge-disjoint matchings M_1^i, \dots, M_m^i of size $(4m+3)$ between every consecutive classes S_i and S_{i+1} . For a pair of terminals (x, y) lying in classes S_i and S_{i+d} (modulo $2m$) at distance d ($1 \leq d \leq m$), choose the edge of M_1^i being adjacent to x and label the other vertex adjacent to it by $p_1(x)$. In step j for $2 \leq j \leq d$ take the edge of M_j^{i+j-1} being adjacent to $p_{j-1}(x)$ and label its other end by $p_j(x)$. Apparently, y and $p_d(x)$ belong to the same class. Phase one ends by assigning a path $P_{xy} = xp_1(x) \dots p_d(x)$ to each (x, y) pair of terminals.

Observe that paths P and P' assigned to and starting at terminals x and x' of the same class do not contain a common vertex as they are given edges of the same matchings in every step. Now assume that edge $e = (x_{i,j}, x_{i+1,k})$ has been utilized by two paths P_1 and P_2 . It means that $e \in M_t^i$ for some $1 \leq t \leq m$ and that P_1 and P_2 must have started in the same class. However, in order to share an edge they also have to share a vertex which contradicts our previous observation. It proves that phase one terminates without edge-collision.

In phase two we finish the linking. For the terminal y initially paired with x and for the endpoint $p_d(x)$ of the path P_{xy} (both vertices lying in S_i for some i) consider the yet unused edges of the bipartite subgraph H_i spanned by S_i and S_{i+1} . As $d_{H_i}(y), d_{H_i}(p_d(x)) \geq 3m+3$, there exists at least $2m+3$ different vertices $z_1, z_2, \dots, z_{2m+3} \in S_{i+1}$ such that $(y, z_k), (p_d(x), z_k) \in E(G), k = 1, 2, \dots, (2m+3)$. Observe that any vertex in S_i is an endpoint of at most $m+1$ paths defined in phase one, hence out of the $(2m+3)$ listed candidates at most $(2m+2)$ could have been assigned to another pair $(y', p_{d'}(x'))$. It means x and y can be joined by the path $xp_1(x) \dots p_d(x)z_iy$ with an appropriate choice of z_i from the list above. That completes the proof.

Proof of Theorem 1

Assume G is a path-pairable graph on n vertices with diameter $d \geq 20$. Let $x, y \in V(G)$ such that $d(x, y) = d$. Define $S_i = \{z \in V(G) : d(x, z) = i\}$ for $i = 0, 1, 2, \dots, d$ and $U_i = \bigcup_{j=0}^i S_j$. Set the notation $s_i = |S_i|$ and $u_i = |U_i|$. Note that there is no edge between any S_i and S_j ($i < j$) classes unless they are consecutive, that is, $j = i+1$. Also, we may assume without loss of generality that vertices belonging to consecutive S_i sets are joined by an edge (adding edges between consecutive classes changes neither the diameter nor the path-pairability property of the graph). If S_d contains vertices in addition to y , move them to S_{d-1} by joining them to every vertex of S_{d-2} as well as to y in case they were not adjacent. Observe, that in our new graph $z \in S_i$ if and only if $d(y, z) = d - i$, $0 \leq i \leq d$. Note again that, while our operation may change the distribution of the vertices among the S_i classes, our newly obtained graph G' has the same diameter as G and is also path-pairable. We could even assume every S_i class to be a complete graph (by adding the necessary edges) without changing the mentioned

properties, though this would be a rather cosmetic operation that would simplify the structure of the graph but would not essentially contribute to the proof.

We introduce the notation S'_i for S_{d-i} and divide our sets into three parts creating left, middle and right segments A, B and C as follows:

$$A = \bigcup_{i=0}^{\lceil \frac{d}{3} \rceil} S_i \quad B = \bigcup_{i=\lceil \frac{d}{3} \rceil+1}^{\lfloor \frac{2d}{3} \rfloor-1} S_i \quad C = \bigcup_{i=\lfloor \frac{2d}{3} \rfloor}^d S_i$$

Our main goal is to give a lower estimate on the size of B . We first prove the following lemmas:

Lemma 1. $s_{2k} + s_{2k+1} \geq k$ as long as $u_{2k+1} \leq \frac{n}{2}$.

Proof. We prove our statement by induction on k . Apparently, $s_0 + s_1 \geq 0$ and $s_2 + s_3 \geq 1$. Observe that the number of edges between S_{2k+2} and S_{2k+3} is at least $u_{2k+1} = \sum_{i=0}^{2k+1} s_i$. Indeed, placing u_{2k+1} terminals in U_{2k+2} and their pairs in $V(G) - U_{2k+2}$ there must be space for at least u_{2k+1} edge-disjoint paths passing from S_{2k+2} to S_{2k+3} . By induction hypothesis $u_{2k+1} \geq \frac{1}{2}k(k+1)$ while the number of edges between the two classes is at most $s_{2k+2}s_{2k+3} \leq \left(\frac{s_{2k+2}+s_{2k+3}}{2}\right)^2$. It yields $\frac{1}{2}k(k+1) \leq \left(\frac{s_{2k+2}+s_{2k+3}}{2}\right)^2$, that is, $s_{2k+2} + s_{2k+3} \geq \sqrt{2k(k+1)} \geq k+1$ if $k \geq 1$. \square

Lemma 2. $|A|, |C| \geq \min(\frac{n}{2}, \frac{d^2}{100})$.

Proof. Assume $|A| < \frac{n}{2}$. Using Lemma 1 we know that $|A| = u_{\lceil \frac{d}{3} \rceil} = s_0 + \dots + s_{\lceil \frac{d}{3} \rceil} \geq 0 + 1 + \dots + \lfloor \frac{\lceil \frac{d}{3} \rceil}{2} \rfloor \geq \frac{d^2}{100}$. By exchanging the role of x and y the same reasoning shows that $|A'| \geq \frac{d^2}{100}$ if $|A'| < \frac{n}{2}$, where $A' = \bigcup_{i=0}^{\lceil \frac{d}{3} \rceil} S'_i$. As $\lceil \frac{d}{3} \rceil + \lfloor \frac{2d}{3} \rfloor = d$, one can easily see that $C = A'$ which completes the proof. \square

If $\lceil \frac{d}{3} \rceil < t < \lfloor \frac{2d}{3} \rfloor$, the number of edges between S_t and S_{t+1} is at least $\min(\frac{n}{2}, \frac{d^2}{100})$. As seen before, placing $\min(\frac{n}{2}, \frac{d^2}{100})$ terminals in A and their pairs in C the set S_t has to be able to bridge $\min(\frac{n}{2}, \frac{d^2}{100})$ disjoint paths to S_{t+1} . The number of crossing edges between these two sets is at most $s_t \cdot s_{t+1}$. It means $\frac{s_t + s_{t+1}}{2} \geq \sqrt{s_t s_{t+1}} \geq \min(\frac{\sqrt{n}}{\sqrt{2}}, \frac{d}{10})$. If $d \geq 16$, then $\lfloor \frac{2d}{3} \rfloor - \lceil \frac{d}{3} \rceil - 2 \geq \frac{d}{3} - \frac{10}{3} \geq 2$ and B contains at least two different S_i sets. That gives us the requested lower bound on $|B|$:

$$|B| \geq \sum_{t=\lceil \frac{d}{3} \rceil+1}^{\lfloor \frac{2d}{3} \rfloor-2} \frac{s_t + s_{t+1}}{2} \geq \left(\frac{d}{3} - \frac{10}{3}\right) \min\left(\frac{\sqrt{n}}{\sqrt{2}}, \frac{d}{10}\right) \geq \frac{d}{6} \min\left(\frac{\sqrt{n}}{\sqrt{2}}, \frac{d}{10}\right) \quad (1)$$

As $|B| \leq n$, our equation proves that $d \leq 6\sqrt{2}\sqrt{n}$.

Summary

We proved that the maximal diameter of a path-pairable graph on n vertices has order of magnitude $O(\sqrt{n})$, that is, $\frac{1}{4}\sqrt{n} < d_{\max}(n) < 6\sqrt{2}\sqrt{n}$ if n is sufficiently large. We mention that in Lemma 1 an even stronger result can be proved about $s_{2k} + s_{2k+1}$.

Lemma 3. Given $\varepsilon > 0$, $s_{2k} + s_{2k+1} \geq (2 - \varepsilon) \cdot k$ holds for sufficiently large k , as long as $u_{2k+1} \leq \frac{n}{2}$.

This result with more careful calculation and elaborate analysis may help reducing the gap between the presented upper and lower bounds.

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